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A Link Between Quantum and Classical Potts Models

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We study ground states of quantum Potts models. We construct ground states of certain d-dimensional quantum models as Gibbs measures of a d-dimensional classical spin system. Our results imply that various phenomena of classical spin systems can also be found in quantum ground states.

KEY WORDS: Quantum Potts model; ground states; Gibbs measure.

1. INTRODUCTION

In this paper, we study the ground states of certain quantum Potts models; these ground states, restricted to multiplication operators, are represented as the Gibbs measures of classical spin systems. Kirkwood and Thomas[(4)] constructed a translationally invariant ground state of quantum Ising models in the weak coupling region. Their idea is based on the observation that the finite-volume ground state of quantum Ising model looks like a Gibbs measure if it is restricted to the subalgebra of observables generated by diagonal matrices. This observation is a corollary of the Perron–Frobenius theorem. Using ideas of ref. 4, we established the uniqueness of the translationally invariant ground state in the infinite-volume limit for weakly coupling systems. See ref. 8.

Weakly coupled quantum systems correspond to the high-temperature phase of the equilibrium state for classical systems. The equivalence of classical and quantum models is different from the transfer matrix method. A d-dimensional quantum system corresponds to a classical system on the same dimensional lattice. In this paper, a more systematic treatment of

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equivalence is done. Here, Gibbs measures are not necessarily translationally invariant nor in the high-temperature phase (translational invariance always mean invariance by any lattice translation).

We introduce quantum spin models on $\mathbb{Z}^d$ to be considered using C^*-algebraic language.[2]

Let q be a positive integer larger than one, and $M_q(\mathbb{C})$ the set of all q by q complex matrices. We consider the C^* algebra A defined by

$$A = \bigotimes_{\mathbb{Z}^d} M_q(\mathbb{C}) \tag{1.1}$$

Let X be an element of $M_q(\mathbb{C})$ and j be in $\mathbb{Z}^d$. By $X^{(j)}$, we denote an element of A with X in the jth component of the tensor product and $\mathbb{1}$ in the other components.

Let α_z be the translation automorphism of A determined via

$$\alpha_z(X^{(j)}) = X^{(j+z)} \tag{1.2}$$

where z and j are in $\mathbb{Z}^d$.

For any subset Λ of $\mathbb{Z}^d$, we define A_Λ as the C^* subalgebra of A generated by all $X^{(j)}$ with j in Λ and X in $M_q(\mathbb{C})$.

We also set

$$(A)_{\text{loc}} = \bigcup_{\Lambda:\text{ finite}} A_\Lambda \tag{1.3}$$

The Hamiltonian of the quantum Potts model is the following formal sum:

$$H = -\sum_{j\in\mathbb{Z}^d} e^{(j)} + \sum_{j\in\mathbb{Z}^d} V_j(U) \tag{1.4}$$

where

$$e = \frac{1}{q}\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} \tag{1.5}$$

and $V_j(U)$ is a (self-adjoint) polynomial of diagonal matrices $U^{(k)}$,

$$U = \begin{pmatrix} 1 & & & & 0 \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ 0 & & & & \omega^{q-1} \end{pmatrix} \tag{1.6}$$

and ω is the qth primitive root of unity.

We also assume translational invariance in the sense that

$$\alpha_k(V_j(U)) = V_{j+k}(U) \tag{1.7}$$

for j and k in $\mathbb{Z}^d$.

The formal sum (1.4) gives rise to a one-parameter group of automorphisms $\gamma_t(\cdot)$:

$$\gamma_t(Q) = e^{itH} Q e^{-itH} \tag{1.8}$$

for Q in A.

A state ω of A is a ground state for H if

$$\frac{1}{i}\frac{d}{dt}\omega(Q^*\gamma_t(Q))\bigg|_{t=0} \geqslant 0 \tag{1.9}$$

for any Q in A_{loc}.

In the next section, we construct the ground state using the Gibbs measure. In this construction, an extremal Gibbs measure gives rise to a pure ground state. See Theorem 2.5. The proofs of results stated in Section 2 are given in Section 3. Section 4 is devoted to proof of the standing assumption of Section 2 in the case that the potential term $V_j(U)$ is small. In Section 5 we give concluding remarks.

2. MAIN RESULTS

In this section we state our results.

First, let Λ be a finite subset of $\mathbb{Z}^d$; the local Hamiltonian H_Λ is determined by

$$H_\Lambda = -\sum_{j\in\Lambda} e^{(j)} + \sum_{j\in\Lambda} V_j(U) \tag{2.1}$$

where the second sum of (2.1) is taken over all j such that V_j is in A_Λ. Then $-H_\Lambda$ acting on $\bigotimes_\Lambda \mathbb{C}^q$ satisfies the assumptions of the Perron–Frobenius theorem. In fact, the nonnegativity of off-diagonal elements is obvious and the irreducibility can be seen as follows.

Let $\hat{H}_\Lambda$ be defined by

$$\hat{H}_\Lambda = -\sum_{j\in\Lambda} e^{(j)} \tag{2.2}$$

Then

$$e^{-\beta\hat{H}_\Lambda} = \prod_{j\in\Lambda} [1 + (e^\beta - 1)e^{(j)}] \tag{2.3}$$

Thus, all the matrix elements of $\exp(-\beta\hat{H}_{\Lambda})$ are positive. This property is not changed if diagonal matrices are added to $\hat{H}_{\Lambda}$.

We consider the basis of $\bigotimes_{\Lambda}\mathbb{C}^q$ of the form.

$$|\sigma\rangle = \bigotimes_{j\in\Lambda} \xi_j \tag{2.4}$$

where

$$\xi_j = \begin{pmatrix} \xi_j^{(1)} \\ \zeta_j^{(2)} \\ \vdots \\ \xi_j^{(q)} \end{pmatrix}$$

and

$$\xi_j^{(k)} = 0, \qquad k \neq k_0(j)$$
$$\xi_j^{(k_0)} = 1$$

The vector $|\sigma\rangle$ of (2.4) is identified with a point of $\mathbb{Z}_q^{\Lambda}$, where $\mathbb{Z}_q = \mathbb{Z}_{|q\mathbb{Z}}$, which is also viewed as $\mathbb{Z}_q = \{\omega^k,\ k = 0, 1, 2,..., q-1\}$. Hence $|\sigma\rangle$ is specified by $\{\omega^{k_j}\}$, $j \in \Lambda$.

By the Perron–Frobenious theorem, the eigenvector of H_{Λ} with the smallest eigenvalue $-|\Lambda| e_{\Lambda}$ can be taken to be a positive vector with respect to the above basis. We set

$$\psi_{\Lambda} = \sum_{\sigma\in\mathbb{Z}_q^{\Lambda}} e^{-h_{(\mathrm{loc})}(\sigma)}\, |\sigma\rangle \tag{2.5}$$

$$H_{\Lambda}\psi_{\Lambda} = -|\Lambda|\, e_{\Lambda}\Psi_{\Lambda} \tag{2.6}$$

where $|\Lambda|$ is the volume of Λ.

Consider the matrix

$$V = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \tag{2.7}$$

Then we define

$$V^{(k)l}\,|\sigma\rangle = |\omega_{(k)}^{l}\sigma\rangle \tag{2.8}$$

that is, the right-hand side is specified with the flip of spin at the site l of σ.

Let $C(\mathbb{Z}_q^A)$ be the set of all continuous functions on $\mathbb{Z}_q^A$ ($\mathbb{Z}_q^A$ is equipped with product topology). To $f(\sigma)$ in $C(\mathbb{Z}_q^A)$, we can associate a vector f via the formula

$$f = \sum_{\sigma} f(\sigma)\, |\sigma\rangle$$

The justification of the notation is due to the fact that

$$V^{(k)-1} f = \sum_{\sigma \in (\mathbb{Z}_q)A} f(\omega^l_{(k)}\sigma)\, |\sigma\rangle \tag{2.9}$$

where we consider $(\mathbb{Z}_q)^A$ as a multiplicative Abelian group and $\omega^l_{(k)}\sigma$ is the product of σ and $\omega^l_{(k)}$ ($=\omega^l$ in the kth component and unity in the others) of this group.

Using this notation, we can rewrite (2.6) as follows:

$$\frac{1}{q} \sum_{j \in A} \left(\sum_{k=0}^{q-1} \exp\left\{ \frac{1}{2} [h_A^{\mathrm{loc}}(\sigma) - h_A^{\mathrm{loc}}(\omega^k_{(j)}\sigma)] \right\} + e_A \right) = \sum_{j \in A} V_j(\sigma) \tag{2.10}$$

Next, we consider the infinite-volume limit. We set

$$B_A = C(\mathbb{Z}_q^A) \tag{2.11a}$$

$$B_{\mathrm{loc}} = \bigcup_{A:\ \mathrm{finite}} B_A \tag{2.11b}$$

$$B = C(\mathbb{Z}_q^{\mathbb{Z}^d}) \tag{2.11c}$$

We can regard B, B_A, and B_{loc} as subalgebras of A generated by diagonal matrices. More explicitly, let E be in $(\mathbb{Z}_q^{\mathbb{Z}^d}))^*$ (the dual group of $(\mathbb{Z}_q^{\mathbb{Z}^{(d)}})$. We set

$$\sigma(E) = \prod_{\{l_j\} = E} \sigma_j^{l_j} \tag{2.12a}$$

$$U(E) = \prod_{\{l_j\} = E} U^{(j) l_j} \tag{2.12b}$$

where E is viewed as a function on E_0 (in $\mathbb{Z}^d$) with value in $\{0, 1, 2, ; q-1\}$. $E_0 = \{j \in \mathbb{Z}^d); l_j \neq 0\}$. Any element $f(\sigma)$ of B_{loc} is a linear combination of monomials $\sigma(E)$,

$$f(\sigma) = \sum_{E \in (\mathbb{Z}_q^{\mathbb{Z}^d})^*} f_E\, \sigma(E) \tag{2.13}$$

B_{loc} is a subalgebra of A_{loc} by map: $f(\sigma) \to f(U)$.

We consider two kind of norms, $\|\cdot\|$ and $\|\cdot\|_\xi$ for B_{loc},

$$\|f(\sigma)\| = \sup_{\sigma \in \mathbb{Z}_q^{\mathbb{Z}^d}} |f_{(\sigma)}| = C^* \text{ norm of } f(U) \tag{2.14}$$

Let $\xi(x)$ be a positive function on the set of real numbers. Then for $F(\sigma)$ of (2.13) we set

$$\|f(\sigma)\|_\xi = \sum_{E \in (\mathbb{Z}_q^{\mathbb{Z}^d})^*} \xi(d(E_0))\, |f_E| \tag{2.15}$$

where E_0 is defined just below (2.12b) and $d(E_0)$ is the diameter of E_0. If $\xi(x)$ satisfies $l \leqslant \xi(x)$, it is easy to show that

$$\|f(\sigma)\| \leqslant \|f(\sigma)\|_1 \leqslant \|f(\sigma)\|_\xi \tag{2.15a}$$

and

$$\|f(\sigma)\, g(\sigma)\|_1 \leqslant \|f(\sigma)\|_1\, \|g(\sigma)\|_1 \tag{2.15b}$$

For $j \in \mathbb{Z}^d$, $k \in \mathbb{Z}_q$, we define

$$\hat{f}(j, k)(\sigma) = \tfrac{1}{2}[f(\sigma) - f(\omega_{(j)}^k \sigma)] \tag{2.16}$$

Assumption 2.1. We assume throughout this section the following:

(i) $\xi(x)$ is a continuous function satisfying $0 < l \leqslant \xi(x)$ and

$$\lim_{r \to \infty} \int_r^\infty \frac{x^{d-1}}{\xi(x)}\, dx = 0 \tag{2.17}$$

(ii) For any j in $\mathbb{Z}$ and k in $\mathbb{Z}_q$, the following limit exists in the norm $\|\cdot\|_\xi$:

$$\lim_{\Lambda \to \infty} \hat{h}_\Lambda^{\mathrm{loc}}(j, k)(\sigma) = \hat{h}(j, k)(\sigma) \tag{2.18}$$

where the limit of Λ is taken in the sense of van Hove.

Remark 2.2.

(i) Assumption 2.1 holds provided $\xi(x) = e^{\delta |x|}$, $\delta > 0$, assuming that $V_j(\sigma)$ satisfies

$$\|V_{(k)}(\sigma)\|_\xi < 2 \log 2 - 1 \tag{2.19}$$

(ii) We use free boundary conditions. Periodic boundary conditions can also be used. The results below need not be altered: $\hat{h}_\Lambda(j, k)(\sigma)$ for different j and k are not independent, because they are determined by a simple function $h(\sigma)$. The same remark is valid for $\hat{h}(j,k)(\sigma)$. More explicitly, consider the expansion by monomials $\sigma(E)$,

$$\hat{h}(j,k)(\sigma)=\sum_{E}\hat{h}_E(j,k)\,\sigma(E) \tag{2.20}$$

This sum is convergent in the norm $\|\cdot\|_\xi$. $\hat{h}_E(j,k)$ is zero except in the case that $\sigma(E)$ contains the term σ_j^k. Furthermore,

$$\hat{h}_{E_1}(j_1k_1)=\hat{h}_{E_2}(j_2k_2) \tag{2.21}$$

if $\sigma(E_2)$ contains the factor $\sigma_{j_1}^{k_1}$ and $\sigma(E_1)$ the factor $\sigma_{j_2}^{k_2}$.

We now define the local Hamiltonian for a classical spin system. Let Λ be a finite subset of $\mathbb{Z}^d$. We define $H_\Lambda(\sigma)$ by

$$H_\Lambda(\sigma)=\sum_{\substack{j\in\Lambda\\ k\in\mathbb{Z}_q}}\sum_{(E_0\in\Lambda)}\frac{\hat{h}_E(j,k)}{|E_0|}\sigma(E) \tag{2.22}$$

Let Γ be in $(\mathbb{Z}_q)^{\mathbb{Z}^d\setminus\Lambda}$. The surface energy $W_\Lambda(\Gamma)(\sigma)$ is determined via

$$W_\Lambda(\Gamma)(\sigma)=\sum_{\substack{j\in\Lambda\\ k\in Z_q}}\sum_{\substack{E_0\cap\Lambda\neq\varnothing\\ E_0\cap\Lambda^c\neq\varnothing}}\frac{\hat{h}_E(j,k)}{|E_0|}\sigma(EV\Gamma) \tag{2.23}$$

where by $(EV\Gamma)$ we mean the classical spin configuration outside Λ. Obviously $W_\Lambda(\Gamma)$ is a well-defined element of B because

$$\|W_\Lambda(\Gamma)(\sigma)\|_\xi\leqslant\sum_{\substack{j\in\Lambda\\ k\in\mathbb{Z}_q}}\|\hat{h}(j,k)\|_\xi<\infty \tag{2.24}$$

Definition 2.3. We define a linear map F from A_{loc} to B by the following equations:

$$F(V(E^a)\,U(E^b))=e^{\hat{h}(E^a)(\sigma)}\,\sigma(E^b) \tag{2.25}$$

$$\hat{h}(E^a)(\sigma)=\tfrac{1}{2}\lim_{\Lambda\to\infty}\{-h_\Lambda(\omega(E^a)J)+h_\Lambda(\sigma)\} \tag{2.26}$$

where E^a and E^b are in $(\mathbb{Z}_q^{\mathbb{Z}^d})^*$, and $w(E)$ is defined in the same way as in (2.12) (ω is the primitive qth root of unity). We again remark that (2.26) is convergent in the norm $\|\cdot\|_\xi$. We now state our main results. Proofs will be given in Section 3.

Theorem 2.4. Suppose Assumption 2.1 is valid. Let $d\mu(\sigma)$ be a Gibbs measure on $\mathbb{Z}_q^{\mathbb{Z}^d}$ associated with the Hamiltonian (2.22). Then there exists a unique ground state ω_μ of the quantum Hamiltonian (2.1) satisfying

$$\omega_\mu(Q) = \int d\mu(\sigma)\, F(Q)(\sigma) \tag{2.27}$$

for any Q in A_{loc}.

See ref. 8 for the definition of Gibbs measure.

Theorem 2.5. Let ω_μ be the ground state associated with the Gibbs measure $d\mu(\sigma)$ of Theorem 2.4. Then ω_μ is pure if and only if $d\mu(\sigma)$ is an extremal Gibbs measure.

Remark 2.6. Thee above results may read in two directions.

(i) Given the quantum model with an explicit form of $V_j(U)$, the theorems tell us several properties of ground states (decay of correlation, nonuniqueness of ground states, etc.).

(ii) Let the classical Hamiltonian $h_A(\sigma)$ be given. Suppose $h_A(\sigma)$ is of finite range. The (2.10) determines $V_j(\sigma)$, hence the quantum Hamiltonian. Some results on Ising and Potts models may be translated into results on quantum models.

We consider the viewpoint (ii). As an example, we give the explicit form of a one-dimensional quantum model associated with the one-dimensional Ising model. The classical Hamiltonian $h(\sigma)$ is given by

$$h(\sigma) = -\beta \sum_{i \in \mathbb{Z}} \sigma_i \sigma_{i+1} \tag{2.28}$$

The quantum Hamiltonian determined via (2.10) is (up to a constant)

$$\begin{aligned} H = & -\sum_{i \in \mathbb{Z}} \sigma_X^{(j)} + 2 \sum_{i \in \mathbb{Z}} (\cosh 2\beta)(\sin 2\beta)\sigma_Z^{(j)}\sigma_Z^{(j+1)} \\ & + \sum_{i \in \mathbb{Z}} (\sin 2\beta)^2 \sigma_Z^{(j)}\sigma_Z^{(j+2)} \end{aligned} \tag{2.29}$$

where σ_x and σ_z are Pauli matrices,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.30}$$

The relation (2.29) gives rise to a generator of a one-parameter group of automorphisms. See Chapter 6.2 of ref. 2.

Considering in the same way the higher-dimensional Potts models gives rise to a quantum Hamiltonian of finite range.

Remark 2.7.

(i) A natural question is to ask about the (non) existence of ground states which do not correspond to any Gibbs measure. In the case of translationally invariant states, see ref. 8.

(ii) We also remark that we can generalize our models, which is done for the $q=2$ case in ref. 8.

3. PROOF OF THEOREMS

Proof of Theorem 2.4 (Proof of Existence of State for A). First we prove the existence of a state satisfying (2.27). We recall that any Gibbs measure is a convex sum of a limit of finite-volume Gibbs measures with suitable boundary conditions (see proposition Cl.2 of ref. 7). We consider the Gibbs measure $d\mu(\cdot)$, which is the limit of $d\mu_{\Gamma_n}$ defined by

$$\mu_{\Gamma_n}(Q)=\frac{\int d\sigma \,\{\exp[-h^{\Gamma_n}(\sigma)]\}\, Q}{Z_n} \tag{3.1}$$

$$\mu_{\Gamma_n}(1)=1 \tag{3.2}$$

$$h^{\Gamma_n}(s)=h_{\Lambda_n}(\sigma)+W_{\Lambda_n}(\Gamma_n)(\sigma) \tag{3.3}$$

where Γ_n is a boundary condition which is specified by the configuration outside the finite region Λ_n.

We may assume that Λ_n is a cube of volume n^d. In the identification of $C(\mathbb{Z}_q^{\mathbb{Z}^d})$ with the subalgebra of A generated by diagonal matrices, the measure $d\mu_{\Gamma_m}$ is the vector state implemented by the vector ξ_m, where

$$\xi_m=\frac{1}{(Z_n)^{1/2}}\left(\sum_{\sigma\in Z_q^{\Lambda_m}}\left\{\exp\left[-\frac{1}{2}h^{\Gamma_m}(\sigma)\right]\right\}|\sigma\rangle\right) \tag{3.4}$$

We fix an extension of the vector state associated with ξ_m to the state of A. Let $\omega_n(\cdot)$ be this extension. Then the following formulas are valid:

$$\omega_n(Q)=w_n(F_m(Q))=\int d\mu_{\Gamma_m}(\sigma)\, F_m(Q)(\sigma) \tag{3.5}$$

where F_m is a linear map from A_{Λ_m} to B satisfying

$$F_m(V(A)\,U(B))=(\exp\{-\tfrac{1}{2}[-h^{\Gamma_m}(\omega(A)\sigma)+h^{\Gamma_m}(\sigma)]\})\,\sigma(B) \tag{3.6}$$

(3.5) can be proved as follows:

$$\omega_n(V(A)\,U(B))$$
$$=\frac{1}{Z_n}\left(\sum_{\sigma,\sigma'}\langle\sigma|\,V(A)\,U(B)\,|\sigma'\rangle\exp\left\{-\frac{1}{2}\left[h^{\Gamma_m}(\sigma)+h^{\Gamma_m}(\sigma')\right]\right\}\right)$$
$$=\frac{1}{Z_n}\left(\sum_{\sigma,\sigma'}\langle\sigma|\,U(B)\,|\sigma'\rangle\exp\left\{-\frac{1}{2}\left[h^{\Gamma_m}(w(A)\sigma)+h^{\Gamma_m}(\sigma')\right]\right\}\right)$$
$$=\frac{1}{Z_n}\left(\sum_{\sigma}\sigma(B)\exp\left\{\frac{1}{2}\left[-h^{\Gamma_m}(w(A)\sigma)+h^{\Gamma_m}(\sigma')\right]\right\}\exp\left[-h^{\Gamma_m}(\sigma)\right]\right) \tag{3.7}$$

[Note that $U(B)$ is diagonal.]

Next we show that $F_m(Q)$ converges to $F(Q)$ for Q in $A_{\rm loc}$. It is easy to see that

$$\lim_{m\to\infty}\tfrac{1}{2}\left[-h^{\Gamma_m}(w(A)\sigma)+h^{\Gamma_m}(\sigma)\right]=\hat{h}(A)(\sigma) \tag{3.8}$$

where $\hat{h}(A)$ is defined in (2.26) and the limit is taken in the norm $\|\cdot\|$.

By (2.15a) and (2.15b), we have

$$\|F_m(V(A)\cup(B))-F(V(A)\cup(B))\|$$
$$\leqslant_e\|\hat{h}(A)(\sigma)\cdot\|_1 e^{\|1/2\left[-h^{\Gamma_m}(w(A)\sigma)+h^{\Gamma_m}(\sigma)\right]-\hat{h}(A)(\sigma)\|_\xi} \tag{3.9}$$

(3.9) implies the convergence we claimed.

Thus, for Q in $A_{\rm loc}$

$$\lim_{n\to\infty}\omega_n(Q)=\lim_{n\to\infty}\int d\mu_n\,F_m(Q)=\int d\mu_n\,F_m(Q)=\int d\mu\,F(Q) \tag{3.10}$$

The existence of ω_μ satisfying (2.27) is proved for the dense set $A_{\rm loc}$ of A. The ω_μ defined by (3.10) can be extended to A. To see this, let $\bar{\omega}$ be a cluster point of $\{\omega_n\}$. Then $\bar{\omega}$ satisfies (2.27). As $A_{\rm loc}$ is dense in A, $\bar{\omega}$ is unique, so $\bar{\omega}$ is the extension we want.

Proof of Theorem 2.4 (Proof of Ground State Property). We next show that the state obtained in the limit (3.10) is the ground state of H. By the uniqueness part of the Perron–Frobenius theorem, ω_n is the ground state for H_n defined by

$$\tilde{H}_m=-\sum_{j\in\Lambda_m}e^{(j)}+\tilde{V}_n \tag{3.11}$$

$$\tilde{V}_n=\sum_{k=1}^{q-1}\sum_{j\in\Lambda_m}\frac{\exp\{\frac{1}{2}[h^{\Gamma_m}(U)-h^{\Gamma_m}(\omega^k_{(j)}U)]\}}{q} \tag{3.12}$$

Note that (3.12) is same as (2.10) except for the boundary surface energy term in $h^{\Gamma_m}(\sigma)$.

We claim that for Q in A_{loc}

$$\lim_{n \to \infty} [\tilde{H}_m, Q] = [H, Q] \tag{3.13}$$

If (3.11) is proved, then

$$\omega_\mu(Q^*[H, Q]) = \lim_{n \to \infty} \omega_m(Q^*[\tilde{H}_m, Q]) \geqslant 0 \tag{3.14}$$

To prove (3.13), it suffices to consider the cases $Q = U^{(j)}$ or $V^{(j)}$. In the first case, (3.13) is obvious, as $V^{(j)}$ commutes with $\tilde{V}_n$ and $V_j(U)$.

We consider $Q = V^{(j)}$.

For m large, we have

$$[\tilde{H}_m, V^{(l)}] = \frac{1}{q}\left(\sum_{k=1}^{q-1} \sum_{j \in \Delta_m} \{\exp[-\tilde{h}^{\Gamma_m}(j, k)(\omega_{(1)}(U))] - \exp[\tilde{h}^{\Gamma_m}(j, k)(U)]\}\right) V^{(l)} \tag{3.15}$$

$$[H_{\Lambda_m}, V^{(l)}] = \frac{1}{q}\left(\sum_{k=1}^{q-1} \sum_{j \in \Lambda_m} \{\exp[\tilde{h}^{\text{loc}} \Lambda_m(j, k)(\omega_{(1)}(U))] - \exp[\tilde{h}^{\text{loc}} \Lambda_m(j, k)(U)]\}\right) V^{(l)} \tag{3.16}$$

where $\tilde{h}^{\Gamma_m}(j, k)(\sigma)$ and $\tilde{h}^{\text{loc}}_{\Lambda_m}(j, k)(\sigma)$ are defined in (2.16) and for $f(\sigma)$ of (2.13).

$$f(\omega_k \sigma) = \sum_{k \in E_0} f_E \sigma_{(E)} + \sum_{l=1}^{q-1} \left[\omega^l \sum_{(j, l) \in E} f_E \sigma(E)\right] \tag{3.17}$$

$f(w_R U)$ is the element of B associated with the function (3.17).

The following lemma implies (3.13).

Lemma 3.1.

$$\lim_{n \to \infty} \left(\sum_{j \in \Lambda_m} \{\exp[\tilde{h}_n^{\#}(j, k)(\omega_1 \sigma)] - \exp[\tilde{h}_n^{\#}(j, k)(\sigma)]\}\right) = \sum_{j \in \mathbb{Z}^d} \{\exp[\tilde{h}(j, k)(\omega_1 \sigma)] - \exp[\tilde{h}(j, k)(\sigma)]\} \tag{3.18}$$

where the limit is taken in $\|\cdot\|_1$ and the right-hand side is convergent in $\|\cdot\|_1$ and

$$\tilde{h}_n^{\#}(j,k)(\sigma)=\tilde{h}_{\Lambda_m}^{\mathrm{loc}}(j,k)(\sigma) \qquad \text{or} \quad \tilde{h}^{\Gamma_m}(j,k)(\sigma)$$

Proof of Lemma 3.1. We consider the case of $\tilde{h}^{\Gamma_m}(j,k)(\sigma)$. Fix a subset Λ of Z^d. Then by (2.15b)

$$\left\| \sum_{j\in\Lambda_m\backslash\Lambda} \{\exp[\tilde{h}^{\Gamma_m}(j,k)(\omega_l\sigma)]-\exp[\tilde{h}^{\Gamma_m}(j,k)(\sigma)]\}\right\|_1$$

$$\leqslant \sum_{j\in\Lambda_m\backslash\Lambda} \exp[\|\tilde{h}^{\Gamma_m}(j,k)(\sigma)\|_1 \, \|\exp[\tilde{h}^{\Gamma_m}(j,k)(\omega_l\sigma)-\tilde{h}^{\Gamma_m}(j,k)(\sigma)]-1\|_1$$

$$\leqslant 2 \sum_{j\in\Lambda_m\backslash\Lambda} \exp[\|\tilde{h}^{\Gamma_m}(j,k)(\sigma)\|_1 \, \|\tilde{h}^{\Gamma_m}(j,k)(\omega_e\sigma)-\tilde{h}^{\Gamma_m}(j,k)(\sigma)\|_1 \quad (3.19)$$

By (2.21)

$$C=\sup_{j,k,n} \|\tilde{h}^{\Gamma_m}(j,k)(\sigma)\|<\infty \tag{3.20}$$

On the other hand, by definition

$$\|\tilde{h}^{\Gamma_m}(j,k)(\omega_l\sigma)-\tilde{h}^{\Gamma_m}(j,k)(\sigma)\|_1 \leqslant 2\sum_{j,l\in E_0} |\tilde{h}_E(j,k)|$$

$$\leqslant \frac{2}{\xi(d(j,l))}\|\tilde{h}(j,k)(\sigma)\|_1 \tag{3.21}$$

where $d(j,l)$ is the distance of j and l.

Thus, if Λ is large,

$$(3.19)\leqslant C'\,\|\tilde{h}(j,k)(\sigma)\|_1 \int_{d(l,\partial\Lambda)}^{\infty} \frac{x^{d-1}}{\xi(d(x,l))}\,dx \tag{3.22}$$

where C' is a constant independent of n.

By (2.17) and (3.22), for any ε positive there exists $\Lambda_{(\varepsilon)}$ uniformly in n such that

$$\left\| \sum_{j\in\Lambda_m\backslash\Lambda_{(\varepsilon)}} \{\exp[\tilde{h}^{\Gamma_m}(j,k)(\omega_l\sigma)]-\exp[\tilde{h}^{\Gamma_m}(j,k)(\sigma)]\}\right\|_1 <\frac{\varepsilon}{2}$$

By the same reason, we have

$$\left\| \sum_{j\in\mathbb{Z}^d\backslash\Lambda_{(\varepsilon)}} \{\exp[\tilde{h}(j,k)(\omega_e\sigma)]-\exp[\tilde{h}(j,k)(\sigma)]\}\right\|_1 <\frac{\varepsilon}{2} \tag{3.23}$$

As is easy to see,

$$\lim_{n \to \infty} \exp[\tilde{h}^{\Gamma_m}(j,k)(\omega_e \sigma)] = \exp[\tilde{h}(j,k)(\sigma)] \tag{3.24}$$

Combined with (3.23), (3.24) leads to (3.18).

The case of $\tilde{h}^{\mathrm{loc}}_{\Lambda_m}(j,k)(\sigma)$ can be treated by the same argument. (QED)

Proof of Theorem 2.5. Let $\{\pi(\cdot), \mathscr{H}, \Omega\}$ be a GNS triple associated with $\omega_\mu(\cdot)$, namely $\pi(\cdot)$ is a morphism for A to $B(\mathscr{H})$, and Ω is the GNS cyclic vector. The center of $\pi(A)''$ is the algebra at infinity m_∞, where

$$m_\infty = \bigcap_{\Lambda:\ \text{finite}} \pi(A_{\Lambda^c})'' \tag{3.25}$$

Let n_∞ be the algebra at infinity for $\pi(B)''$,

$$n_\infty = \bigcap_{\Lambda:\ \text{finite}} \pi(B_{\Lambda^c})'' \tag{3.26}$$

Obviously n_∞ is the subalgebra of m_∞; the Gibbs measure $d\mu$ is extremal if and only if n_∞ is trivial.[6] We now give a proof of $n_\infty = m_\infty$.

Lemma 3.2. Let Q be in A_{loc}. Then

$$\pi(Q)\Omega = \pi(F(Q^*)^*)\Omega \tag{3.27}$$

Proof of (3.27). We note that if Q_2 is in B_{loc},

$$F(Q_1 Q_2) = F(Q_1) Q_2 \tag{3.28}$$

[See the definition of $F(i)$.] Thus, for Q_2 in B_{loc}

$$\omega_\mu(Q_1^* Q_2) = \omega_\mu(F(Q_1^*) Q_2) = (\pi(F(Q_1^*))^* \Omega, \pi(Q_2)\Omega) \tag{3.29}$$

As $\pi(B)''$ is identified with $L^\infty(\mathbb{Z}_q^{\mathbb{Z}^d})$ in $L^2(d\mu)$, Ω is cyclic and separating for $\pi(B)''$ in $\mathscr{H}$.

Thus, (3.29) leads to (3.27). (QED)

Let B_1 be the completion of B_{loc} by the norm $\|\cdot\|_1$. Then $F(\cdot)$ is a map from A_{loc} to B_1. Fix a bounded set Λ of $\mathbb{Z}^d$. Consider the action $Ad(V(E))$ of E in $\mathbb{Z}_q^\Lambda$ on B_1. Then, it is easy to see that $Ad(V(E))$ leaves B_1 invariant and

$$\|Ad(V(E))(Q)\|_1 = \|Q\|_1 \quad \text{for} \quad Q \text{ in } B_1 \tag{3.30}$$

Lemma 3.3. Let dE be the normalized Haar measure of the finite group Z_q^A. For Q in B_1 we set

$$E_A(Q) = \int dE\, Ad(V(E))(Q) \tag{3.31}$$

Then E_A maps B_1 onto $(B_{A^c})_1$, where $(B_{A^c})_1$ is the completion of $B_{A^c} \cap B_{\text{loc}}$ by $\|\cdot\|_1$.

Proof of Lemma 3.3. This is obvious from (2.13), which is convergent if f is in B_1. (QED)

Lemma 3.4. $n_\infty = m_\infty$.

Proof of Lemma 3.4. As note above, $\pi(B)''$ is maximally Abelian, so $m_\infty \sqsubseteq \pi(B)''$. Let Q in m_∞. Let A_m be an increasing sequence of bounded subsets in $\mathbb{Z}^d$ and Q_m be in B_{A_n} such that $\pi(Q_m)$ converges to Q in strong operator topology. As $\mathbb{Z}_q^A$ is a finite group,

$$\lim_{n \to \infty} \pi(\varepsilon_A(Q_m)) = \int dg\, Ad(\pi(V(E)))(Q) = Q \tag{3.32}$$

(3.32) tells us that Q is in $\pi(B_{A^c})''$. (QED)

4. WEAK COUPLING EXPANSION

In this section, we show that Assumption 2.1 is valid if the potential term $V_j(U)$ is sufficiently small. The proofs are the same as those of ref. 4. We explain these here, as we consider a slightly Hamiltonian.

Proposition 4.1. Let $\xi = e^{|x|}$. If $\|V_j(\sigma)\|_\xi < 2\log 2 - 1$, then (2.18) is valid and

$$\|\hat{h}(j, k)(\sigma)\|_\xi < \ln \to \log 2 \tag{4.1}$$

Proof of Proposition 4.1. The proof is essentially the same as in ref. 4. We sketch the proof. Let $H_A(\lambda)$ be the local Hamiltonian defined by

$$H_A(\lambda) = -\sum_{j \in A} e^{(j)} + \lambda \sum_{j \in A} V_j(U) \tag{4.2}$$

We now use the periodic boundary condition as in refs. 4 and 8. As we already mentioned, this change is not essential and the proofs of Theorem 2.4 and 2.5 are still valid.

Let $A^{(1)}$, $A^{(2)}$ be in $(\mathbb{Z}_q^{\mathbb{Z}^d})^*$ and $A^{(1)}$, $A^{(2)}$ be the group multiplication. Then

$$d(A^{(1)}, A^{(2)}) \leqslant d(A^{(1)}) + d(A^{(2)}) \tag{4.3}$$

where $d(A)$ is the diameter of the set A_0 [see the equation just after (2.12b)].

By (4.3) and (2.15) it is possible to show

$$\| f(\sigma)\, g(\sigma)\|_\xi \leqslant \|f(\sigma)\|_\xi \; \|g(\sigma)\|_\xi \tag{4.4}$$

By the uniqueness of the Perron–Frobenious vector and analytic perturbation theory of eigenvalues of matrices, $h_A^{\rm loc} = h_A(\lambda)$ of (2.10) is an analytic function of λ in a neighborhood of λ. We show an estimate of the radius of convergence (which is uniform in A) as well as (2.18).

Consider the expansions

$$e_A(\lambda) = \sum_{n=0}^{\infty} e_A^{(n)} \lambda^n \tag{4.5a}$$

$$h_A(\lambda)(\sigma) = \sum_{n=0}^{\infty} h_A^{(n)}(\sigma) \lambda^n \tag{4.5b}$$

Then (2.10) leads to

$$\sum_{j \in A} \left\{ \left[\frac{1}{q} \sum_{k=1}^{q-1} \tilde{h}_A^{(1)}(j, k)(\sigma) \right] - V_j(\sigma) + e_A^{(1)} \right\} = 0 \tag{4.6a}$$

$$\sum_{j \in A} \left[\frac{1}{q} \sum_{k=1}^{q-1} \tilde{h}_A^{(n)}(j, k) + P_n(\tilde{h}_A^{(1)}(j, k); \tilde{h}_A^{(n-1)}(j, k)) + e_A^{(n)} \right] = 0 \tag{4.6b}$$

if $n \geqslant 2$, where $P_n(x_1,..., x_{n-1})$ is a positive polynomial defined by

$$\exp\left(\sum_{k=1}^{\infty} x_h \lambda^k \right) = 1 + \sum_{k=1}^{\infty} [x_k + P_k(x_1,..., x_{k-1})] \lambda^k \tag{4.7}$$

If we use the monomial expansion of $h_A^{(n)}$ of the form

$$h_A^{(n)}(\sigma) = \sum_E J_{A,E}^{(n)} \sigma(E) \tag{4.8}$$

then by $1 + w + w^2 + \cdots + w^{q-1} = 0$

$$\begin{aligned}\sum_{j \in A} \frac{1}{q} \sum_{k=1}^{q-1} \tilde{h}_A^{(n)}(j, k)(\sigma) &= \frac{1}{2} \sum_{j \in A} \left[\sum_{j \in E_0} J_{A,E}^{(n)} \sigma(E) \right] \\ &= \sum_E |E_0|\, J_{A,E}^{(n)} \sigma(E)\end{aligned} \tag{4.9}$$

Thus (4.6) defines $h_{\Lambda}^{(n)}(\sigma)$ up to the constant term $J_{\Lambda,\phi}^{(n)}$. The constant term fixes the normalization of the ground-state vector, so it is irrelevant to our analysis.

By definition, we have

$$\|\tilde{h}_{\Lambda}^{(n)}(j,k)(\sigma)\|_{\xi} \leqslant \sum_{j \in E_0} |J_{\Lambda,E}^{(n)}| \, \xi(d(E)) \tag{4.10}$$

We define $J_{\Lambda,E}^{(n)}(\sigma)$ by the equation

$$J_{\Lambda,E}^{(n)}(\sigma) = \sum_{j \in E_0} J_{\Lambda,E}^{(n)} \sigma(E) \tag{4.11}$$

By translational invariance [or periodicity of $H_{\lambda}(\lambda)$], we have

$$\left\| \sum_j J_{\Lambda,j}^{(n)}(\sigma) \right\|_{\xi} = |\Lambda| \, \|J_{\Lambda,j}^{(n)}(\sigma)\|_{\xi} \tag{4.12}$$

Due to (4.4), (4.6), and (4.10) and the positivity of coefficients of $P_n(x_1,...)$ we have

$$\|J_{\Lambda,j}^{(1)}(\sigma)\|_{\xi} \leqslant \|V_j(\sigma)\|_{\xi} \tag{4.13a}$$

$$\|J_{\Lambda,j}^{(n)}(\sigma)\|_{\xi} \leqslant P_n \|J_{\Lambda,j}^{(1)}(\sigma)\|_{\xi} \cdots \|J_{\Lambda,j}^{(n-1)}(\sigma)\|_{\xi} \tag{4.13b}$$

Let C_n be a sequence of positive numbers determined recursively by

$$C_1 = \|V_j(\sigma)\|_{\xi} \tag{4.14a}$$

$$C_n = P_n(C_1,..., C_{n-1}) \tag{4.14b}$$

By the result of ref. 5, the following function $I(\lambda)$ converges if $|\lambda| \, \|V_j(\sigma)\|_{\xi} < 2 \log 2 - 1$:

$$I(\lambda) = \sum_{k=1}^{\infty} C_k \lambda^k \tag{4.15}$$

Furthermore, it can be shown[3] that

$$I(\lambda) \leqslant \log 2 \tag{4.16}$$

It is easy to prove the following inequality by induction:

$$\|J_{\Lambda,j}^{(n)}(\sigma)\|_{\xi} \leqslant C_n \tag{4.17}$$

Thus,

$$\sum \|\tilde{h}_{\Lambda}^{(n)}(j,k)(\sigma)\|_{\xi} \leqslant \log 2 \tag{4.18}$$

and (4.5b) is convergent uniformly in Λ provided that

$$|\lambda| \, \|V_j(\sigma)\|_\xi < 2 \log 2 - 1 \tag{4.19}$$

To prove the convergence of (2.18), note that if Λ is contained in Λ',

$$J^{(n)}_{\Lambda,j}(\sigma) = J^{(n)}_{\Lambda',j}(\sigma) \tag{4.20}$$

for $n \leqslant (1/r)\, d(\Lambda)$, where r is the range potential of $V_j(\sigma)$.

Equation (4.20) is a consequence of translational invariance. Equation (2.6) is effectively the same equation up to order n less than $(1/r)\, d(\Lambda)$.

Set

$$J^{(\Lambda)(\sigma)}_{\Lambda,j} = \sum_{n=1}^{\infty} J^{(n)}_{\Lambda,j}(\sigma)\lambda^n \tag{4.21}$$

Then, $\lim_{n\to\infty} J^{(n)}_{\Lambda,j}(\sigma)$ exists in $\|\cdot\|_\xi$ by Eq. (4.20). Due to (4.8) and (4.11), we have

$$\|\tilde{h}_\Lambda(j,k)(\sigma) - \tilde{h}_{\Lambda^1}(j,k)(\sigma)\|_\xi \leqslant \|J^{(n)}_{\Lambda,j}(\lambda)(\sigma) - J^{(n)}_{\Lambda',j}(\lambda)(\sigma)\|_\xi \tag{4.22}$$

(4.22) implies (2.6). (QED)

5. REMARKS

Our results show that ground states of quantum Potts models may be studied by techniques of classical spin models. However, results of ref. 1 suggest that Assumption 2.1 corresponds only to the weak coupling region. The absence of phase transition for spin systems with short-range interactions in one dimension is well known. For the exactly solved XY model, the existence of a ground-state phase transition has been established. Thus, if $V_j(U)$ in (2.1) is large, the classical spin system associated with the ground state of H in (2.1) has a long-range potential.

In special cases, we may also study the strong coupling region. For example,

$$H_{\text{Ising}} = -\sum_{j\in Z^d} \sigma_x^{(j)} + \lambda \sum_{|j-j'|=1} \sigma_z^{(j)}\sigma_z^{(j)} \tag{5.1}$$

$$H_{XY}(\gamma) = -\sum_{|j-j'|=1} \sigma_x^{(j)}\sigma_x^{(j')}(1 + \alpha\sigma_z^{(j)}\sigma_z^{(j')}) \tag{5.2}$$

where σ_x and σ_z are Pauli spin matrices.

In (5.1) the first term may be regarded as a perturbation. If we consider the system restricted to the $\mathbb{Z}_2$-invariant part, we can use our approach. As a consequence, we can prove the uniqueness of $\mathbb{Z}_2$-invariant, translationally-invariant ground states if λ is large in (5.1) and if α is small in (5.2).

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REFERENCES

1. H. Araki and T. Matsui, Ground states of the *XY* model, *Commun. Math. Phys.* **101**:213–245 (1985).
2. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer, Berlin, 1981).
3. R. Hagedorn and J. Rafelski, Analytic structure and explicit solution of an important implicit equation, *Commun. Math. Phys.* **83**:538–578 (1982).
4. J. R. Kirkwood and L. Thomas, Expansions and phase transitions for the ground state of quantum Ising lattice systems, *Commun. Math. Phys.* **88**:569–580 (1983).
5. M. Kohmoto, den Nijs, and L. P. Kadanoff, Hamiltonian studies of the $d=2$ Ashkin–Teller model, *Phys. Rev. B* **24**:5229 (1981).
6. O. E. Lanford and D. Ruelle, Observable at infinity and states with short range correlations in statistical mechanics, *Commun. Math. Phys.* **13**:194–215 (1969).
7. O. E. Lanford, *Lecture Notes in Physics*, Vol. 20 (Springer, Berlin, 1973).
8. T. Matsui, Uniqueness of translationally invariant ground states in quantum spin system, *Commun. Math. Phys.* **126**:453–467 (1990).
9. von Gehlen and V. Rittenberg, Operator content of the three-state Potts quantum chain, *J. Phys. A* **19**:L625–629 (1986).